AIP | Journal of Applied Physics

Representation of a Function by Its Line Integrals, with Some Radiological Applications

A. M. Cormack

Citation: J. Appl. Phys. **34**, 2722 (1963); doi: 10.1063/1.1729798
View online: http://dx.doi.org/10.1063/1.1729798
View Table of Contents: http://jap.aip.org/resource/1/JAPIAU/v34/i9
Published by the American Institute of Physics.

Representation of a Function by Its Line Integrals, with Some Radiological Applications

A. M. CORMACK
Physics Department, Tufts University, Medford, Massachusetts

(Received 28 January 1963; in final form 26 April 1963)

A method is given of finding a real function in a finite region of a plane given its line integrals along all straight lines intersecting the region. The solution found is applicable to three problems of interest for precise radiology and radiotherapy: (1) the determination of a variable x-ray absorption coefficient in two dimensions; (2) the determination of the distribution of positron annihilations when there is an inhomogeneous distribution of the positron emitter in matter; and (3) the determination of a variable density of matter with constant chemical composition, using the energy loss of charged particles in the matter.

I. INTRODUCTION

THE exponential absorption of a parallel beam of x or gamma rays passing through homogeneous materials has been known and used quantitatively for a long time, but the problem of the quantitative determination of the variable absorption coefficient in inhomogeneous media has received little or no attention. To be sure, all radiography depends on the variation of the absorption coefficient of a medium in space, but the correct interpretation of radiographs depends on the art of the radiographer rather than on measurements.

While the problem of determining such variable absorption coefficients is interesting in itself, it also has an important application in any attempt at precise radiotherapy. The object of the radiotherapist is to direct an external beam, or beams, of x rays at a patient in such a way that a particular region of the patient's interior receives a known dose of radiation, while other parts of the patient receive as small a dose as possible. It is clearly necessary to know the absorption coefficients of the patient's various kinds of bone and tissue in order to make a precise estimate of the dosage received at any point of his interior, and it is equally clear that such information may only be obtained from measurements made exterior to the patient.

It is sufficient to consider the problem in two dimensions, since, if a solution can be found for two dimensions, the three-dimensional case may be solved by considering it to be a succession of two-dimensional layers.

The problem may be quantitatively formulated as follows. Let D be a finite, two-dimensional domain in which there is absorbing material characterized by a linear absorption coefficient g which varies from point to point in D and is zero outside D. Although $g \geqslant 0$, it is convenient to allow it to be negative for purposes of discussion. Suppose a parallel, indefinitely thin beam of monoenergetic gamma rays traverses D along a straight line L, and that the intensity of the beam incident on D is I_0, and the intensity of the beam emerging from D is I. Then

$$I = I_0 \exp\left[-\int_L g(s)ds\right], \qquad (1)$$

where the L under the integral indicates that the integral is to be evaluated along all of L in D, and s is a measure of distance along L. If $f_L = \ln(I_0/I)$, then

$$f_L = \int_L g(s)ds. \qquad (2)$$

The problem is to find g, knowing the line integrals f_L for a number of lines L which intersect D.

One might think that a suitable way of finding g (suggested by taking two radiographs in directions at right angles to each other) would be by measuring f_L along two sets of parallel lines at right angles to each other. That this will not do may be seen as follows. Suppose that D is a square, and that this square is subdivided into n^2 equal smaller squares. Suppose also that the directions along which the line integrals are evaluated are parallel to the sides of the square. If in each small square, g assumes its average value in that square, then the line integrals in Eq. (2) may be approximated by the sums of the average values of g in the rows and columns of smaller squares. This procedure yields $2n$ values of f_L, and Eq. (2) reduces to a set of $2n$ equations in the n^2 average values of g in the small squares. Furthermore, the sum of the n values of f_L parallel to a side must be equal to the sum of the n values parallel to a side at right angles, hence, only $2n-1$ of the equations are independent. The problem is thus indeterminate except for the trivial case $n=1$. This indeterminancy is illustrated in the case $n=2$, which yields three independent equations in four unknowns. Consider a square to be subdivided into four equal smaller squares, and suppose $g=-1$ in two diagonally opposite squares, and $g=1$ in the other two. Then the line integrals (in the exact sense) along all lines parallel to the sides are zero, and furthermore, they are unaffected by the interchange of the above values of g, even though $g \neq 0$.

One is naturally led to consider measuring f_L along lines parallel to a larger number of different directions. Here again one can construct examples in which the f_L are zero along all lines parallel to a number of directions even though $g \neq 0$. For example, consider g to be confined to a circle, and let it have the value $A \cos n\theta$ in the circle, where $n>0$. Then the line integrals of g are zero along all lines perpendicular to the directions $\theta=(2m+1)\pi/2n$, where $m=0, 1, 2\cdots, 2n-1$. However, the f_L is not zero along lines other than those specified.

These considerations suggest that if a solution to the problem can be found at all, it must be sought by considering f_L along all lines intersecting D and then seeing whether an approximate solution may be found by considering only a finite number of lines, so that the problem may be tractable in practice. The following problem is thus considered. An unknown, suitably restricted, real function g exists in a finite two-dimensional domain D and is zero outside D, and the line integrals of g along all straight lines intersecting D are known. Is it possible to determine g? One would think that this problem would be a standard part of the nineteenth century mathematical repertoire, but the author has found no reference to it in standard works.

The same mathematical problem occurs in two other radiological procedures. In the first,[1] to locate tumors in say the head, the patient is given Cu^{64} or some other positron emitter. This tends to concentrate in abnormal tissue in the brain but unfortunately it also tends to concentrate in the muscles of the face and neck. The positrons annihilate and pairs of annihilation photons are emitted in very nearly opposite directions, the very small deviations from colinearity being caused by the momentum distributions of the electrons and positrons which annihilate. The annihilation photons are observed by two counters in coincidence, located on opposite sides of the head. The coincidence rate is determined by the distribution of the rate of annihilation along the line joining the counters, and by the absorption of annihilation radiation along that line. The latter factor can be found by measurements made external to the head. It can be shown that if g is the rate at which annihilations occur along the line L joining the counters, and if f_L is the observed coincidence rate corrected for absorption, f_L and g are related by Eq. (2).

The next application of the solution of Eq. (2) concerns the recent use of the peak in the Bragg curve for the ionization caused by protons, to produce small regions of high ionization in tissue.[2] The radiotherapist is confronted with the problem of determining the energy of the incident protons necessary to produce the high ionization at just the right place, and this requires knowing the variable specific ionization of the tissue through which the protons must pass. This problem is very complicated, for the rate of energy loss of protons depends on both their energy and the chemical composition of the material in which they are slowing down. However, the problem is the same as the one considered for x rays, if one can assume that the tissue varies in density but not in chemical composition. In this case, if a fine beam of protons passes along L, and their energies incident on, and emergent from, D are known, the number of g/cm^2 of material along L can be found from the range-energy relation for the material. This can be represented by f_L, and the relation between f_L and the local density, g, is again given by Eq. (2). The applicability to this case of the results given below seems more difficult than in the above two cases. For one thing, bone and tissue have variable chemical composition so the procedure could be at best approximate, and for another, the difference between the energies of the incident and emergent protons is difficult to measure accurately because of the finite spread in energy of any incident beam and because of straggling.

2. FORMULATION OF THE PROBLEM AS A SET OF INTEGRAL EQUATIONS

It is sufficient to consider the domain of g to be a circle of radius R. For one thing, any finite domain may be contained in a circle, and, as is later seen, the circle has a special significance for the method by which the solution is obtained. For simplicity the circle is taken to have unit radius. The origin of polar coordinates (r,θ) may be taken to be at the center of the circle and we may write $g=g(r,\theta)$. The line L along which g is to be inegrated may be defined by the parameters (p,ϕ), where p is the perpendicular distance from the origin to the line L, and ϕ is the angle which the normal to L makes with the $\theta=0$ axis, and f_L may be considered to be a function of the polar coordinates (p,ϕ). The unique property of the circular domain is now apparent: in the (p,ϕ) plane, the domain of f is the same as the domain of g, namely a circle of unit radius. Equation (2) may now be written

$$f(p,\phi)=\int_{L(p,\phi)} g(r,\theta)ds. \tag{3}$$

Equation (3) is an integral equation in two variables, but it may be reduced to a set of integral equations in one variable as follows. Suppose that g is finite, single-valued and continuous, except along a finite number of arcs in the circle, then it may be expanded in a Fourier series:

$$g(r,\theta)=\sum_{n=-\infty}^{+\infty} g_n(r)e^{in\theta}, \tag{4}$$

where

$$g_n(r)=\frac{1}{2\pi}\int_0^{2\pi} g(r,\theta)e^{-in\theta}d\theta. \tag{5}$$

Consider the contribution df to $f(p,\phi)$ from two equal elements of arc ds of the line $L(p,\phi)$. If the elements of arc are on opposite sides of the point (p,ϕ) and

[1] F. W. Wrenn, M. L. Good, and P. Handler, Science **113**, 525 (1951); G. L. Brownell and W. H. Sweet, Nucleonics **11**, 40 (1953).
[2] R. R. Wilson, Radiology **47**, 487 (1946); B. Larsson, L. Leksell, B. Rexed, P. Saurander, W. Mair, and B. Andersson, Nature **182**, 1222 (1958).

are equally spaced from it, then

$$df=\sum_n \left[g_n(r)e^{in\theta}+g_n(r)e^{in(2\phi-\theta)}\right]ds,$$

since if one element is at the point (r,θ) the other must be at $(r, 2\phi-\theta)$. Thus

$$df=2\sum_n g_n(r)e^{in\phi}\cos[n(\theta-\phi)].$$

Then since $s=(r^2-p^2)^{\frac{1}{2}}$, and $\theta-\phi=\cos^{-1}(p/r)$,

$$f(p,\phi)=\sum_n e^{in\phi}2\int_p^1\frac{g_n(r)\cos[n\cos^{-1}(p/r)]rdr}{(r^2-p^2)^{\frac{1}{2}}}. \quad (6)$$

Now $f(p,\phi)$ is a function of polar coordinates (p,ϕ) in the unit circle, so it may be expanded in a Fourier series:

$$f(p,\phi)=\sum_{n=-\infty}^{+\infty} f_n(p)e^{in\phi}, \quad (7)$$

where

$$f_n(p)=\frac{1}{2\pi}\int_0^{2\pi}f(p,\phi)e^{-in\phi}d\phi. \quad (8)$$

Comparing Eqs. (6) and (7) it is seen that the problem separates, and its solution depends on the solution of the set of one-dimensional integral equations:

$$f_n(p)=2\int_p^1\frac{g_n(r)\cos[n\cos^{-1}(p/r)]rdr}{(r^2-p^2)^{\frac{1}{2}}}. \quad (9)$$

$\cos(n\cos^{-1}x)$ is a polynomial of degree n in x known as the Tschebycheff polynomial of the first kind. It is denoted by $T_n(x)$, and Eq. (9) is written

$$f_n(p)=2\int_p^1\frac{g_n(r)T_n(p/r)rdr}{(r^2-p^2)^{\frac{1}{2}}}, \quad (10)$$

because in the solution of this equation, $T_n(x)$ is used for $x>1$ for which the cosine representation is inconvenient.

Some properties of the $T_n(x)$ are given below. A complete account of them can be found in Tricomi.[3] (i) $T_n(1)=1$. (ii) $T_n(x)$ has n distinct zeros, say $x_1, x_2, \cdots, x_n$, which satisfy the inequality $-1<x_1<x_2,\cdots,<x_n<1$, where $x_n=\cos(\pi/2n)$. (iii) The $T_n(x)$ are orthogonal in $(-1, 1)$ with weight function $(1-x^2)^{-\frac{1}{2}}$. (iv) $T_n(-x)=(-1)^nT_n(x)$.

3. SOME PROPERTIES OF THE $f_n(p)$

The restrictions placed on $g(r,\theta)$ and hence on the $g_n(r)$, imply restrictions on the behavior of the $f_n(p)$. Some of these are interesting in that they enable one to anticipate the behavior of the $f_n(p)$ in experimental situations, but others are of greater importance since they bear on the uniqueness of the solution of the problem.

(a) If $g_n(r)$ is bounded and piecewise continuous, $f_n(p)$ is bounded and continuous.

(b) $f_n(p)$ is uniquely determined by $g_n(r)$ through Eq. (10).

Suppose that there were two different functions $g_{n1}(r)$ and $g_{n2}(r)$ which both gave rise to the same $f_n(p)$. Then their difference $G_n(r)$ would satisfy

$$\int_p^1\frac{G_n(r)T_n(p/r)rdr}{(r^2-p^2)^{\frac{1}{2}}}=0. \quad (11)$$

By properties (i) and (ii) above, $T_n(x)\geqslant 0$ for $x_n\leqslant x\leqslant 1$. Consider Eq. (11) for $p\geqslant x_n$. Then $x_n\leqslant p\leqslant (p/r)\leqslant 1$, so $T_n(p/r)\geqslant 0$ and the integrand is positive apart from $G_n(r)$. Hence, the only continuous solution of Eq. (11) in $x_n\leqslant p\leqslant 1$ is $G_n(r)=0$. Thus it is only necessary to consider

$$\int_p^{x_n}\frac{G_n(r)T_n(p/r)rdr}{(r^2-p^2)^{\frac{1}{2}}}=0. \quad (12)$$

Now let $x_n^2\leqslant p\leqslant x_n$. Then $x_n\leqslant (p/x_n)\leqslant (p/r)\leqslant 1$. Hence again $T_n(p/r)\geqslant 0$ for $x_n^2\leqslant p\leqslant x_n$, and Eq. (12) only has the solution $G_n(r)=0$ for $x_n^2\leqslant r\leqslant x_n$. Repetition of this process m times shows that the only solution of Eq. (11) in $x_n^m\leqslant r\leqslant 1$ is $G_n(r)=0$. Let $m\rightarrow\infty$, then $x_n^m\rightarrow 0$ since $x_n<1$. Hence the only solution of Eq. (11) is $G_n(r)=0$ in $0<r\leqslant 1$, and the $f_n(p)$ are uniquely determined by the $g_n(r)$ except at $p=0$.

(c) $|f_n(p)|\leqslant 2M_n(1-p^2)^{\frac{1}{2}}$ for $x_n\leqslant p\leqslant 1$, where M_n is some positive constant.

It was proved above that $T_n(p/r)\geqslant 0$ if $x_n\leqslant p\leqslant 1$. It is also decreasing in this range. Then, since $g_n(r)$ is bounded, $|g_n(r)|\leqslant M_n$ where M_n is some positive constant, so

$$|f_n(p)|\leqslant 2M_nT_n(1)\int_p^1\frac{rdr}{(r^2-p^2)^{\frac{1}{2}}}=2M_n(1-p^2)^{\frac{1}{2}}.$$

In particular, $f_n(1)=0$.

(d) $f_{2n+1}(0)=0$. $f_{2n}(0)=(-1)^n2\int_0^1g_n(r)dr$.

Consider $f(0,\phi)$ for $g(r,\theta)=g_n(r)\cos n\theta$ or $g_n(r)\sin n\theta$ and the result can be seen from the symmetry of the problem.

(e) $f(0,\phi)=f(0,\phi+\pi)$.

This result must hold since nowhere in the formulation of the problem has a positive direction for the evaluation of the line integral been assigned. The formal demonstration is:

$$f(0,\phi+\pi)=\sum_n f_n(0)(-1)^ne^{in\phi}=\sum_n f_n(0)e^{in\phi}=f(0,\phi),$$

since $f_{2n+1}(0)=0$.

(f) $f_{2n}(p)$ and $f_{2n+1}(p)$ each change sign at least n times in $0<p<1$.

[3] F. G. Tricomi, *Vorlesungen Uber Orthogonal Reihen* (Springer-Verlag, Berlin, 1955).

Multiply both sides of Eq. (10) by p^k and integrate from $p=0$ to $p=1$:

$$\int_0^1 f_n(p)p^k dp = 2\int_0^1 p^k dp \int_p^1 \frac{g_n(r)T_n(p/r)rdr}{(r^2-p^2)^{\frac{1}{2}}}$$

$$= 2\int_0^1 g_n(r)rdr \int_0^r \frac{p^k T_n(p/r)dp}{(r^2-p^2)^{\frac{1}{2}}}$$

$$= 2\int_0^1 g_n(r)r^{k+1}dr \int_0^1 \frac{t^k T_n(t)dt}{(1-t^2)^{\frac{1}{2}}}.$$

If n is even (odd), $T_n(x)$ is orthogonal in (0,1) to any even (odd) polynomial of degree less than n because of the orthogonality of the $T_n(x)$ in $(-1,1)$ and because of their symmetry [property (iv)]. In particular, depending on the evenness or oddness of n and k, $T_n(x)$ is orthogonal to t^k, so for $k<n$

$$\int_0^1 f_{2n}(p)p^{2k}dp=0, \quad \int_0^1 f_{2n+1}(p)p^{2k+1}dp=0. \qquad (13)$$

With Eq. (13) established, the proof is very like the proof of the number of zeros of orthogonal polynomials. Suppose that $n>0$, then $\int_0^1 f_{2n}(p)dp=0$. Hence $f_{2n}(p)$ must change sign at least once in (0,1). Suppose that it changes sign m times, at points $p_1, p_2, p_3, \cdots, p_m$, and consider the integral

$$I=\int_0^1 f_{2n}(p)(p^2-p_1^2)(p^2-p_2^2)\cdots(p^2-p_m^2)dp.$$

The integrand is either everwhere positive except at the zeros, or it is everywhere negative except at the zeros, that is $|I|>0$. But

$$I=a_1\int_0^1 f_{2n}(p)p^{2m}dp + a_2\int_0^1 f_{2n}(p)p^{2m-2}dp+\cdots+a_m\int_0^1 f_{2n}(p)dp,$$

where $a_1, a_2, \cdots, a_m$ are some constants, and from Eq. (13), the integrals are zero if $m<n$. This contradicts the assumption that $|I|\geqslant 0$, unless $m\geqslant n$. Hence, $f_{2n}(p)$ changes sign at least n times in $0<p<1$. A similar proof holds for $f_{2n+1}(p)$, since if $\int_0^1 pf_{2n+1}(p)dp=0$, $f_{2n+1}(p)$ must change sign at least once. Consideration of the integral

$$I=\int_0^1 f_{2n+1}(p)(p^2-p_1^2)(p^2-p_2^2)\cdots(p^2-p_m^2)pdp$$

then yields the desired result.

4. SOLUTION OF THE EQUATIONS

Multiply both sides of Eq. (10) by $T_n(p/z)(z/p)\times(p^2-z^2)^{-\frac{1}{2}}$, integrate from $p=z$ to $p=1$ and interchange the order of integration on the right hand side:

$$\int_z^1 \frac{zT_n(p/z)f_n(p)dp}{p(p^2-z^2)^{\frac{1}{2}}} = 2\int_z^1 g_n(r)dr\int_z^r \frac{rzT_n(p/z)T_n(p/r)dp}{p(r^2-p^2)^{\frac{1}{2}}(p^2-z^2)^{\frac{1}{2}}}. \qquad (14)$$

If the p integration on the right is denoted by $I_n(r,z)$, where

$$I_n(r,z)=rz\int_z^r \frac{T_n(p/z)T_n(p/r)dp}{(r^2-p^2)^{\frac{1}{2}}(p^2-z^2)^{\frac{1}{2}}p}, \qquad (15)$$

then it can be shown that $I_{n+1}=I_{n-1}$, $I_0=I_1=\pi/2$, so

$$I_n(r,z)=\pi/2. \qquad (16)$$

Hence, Eq. (14) becomes

$$\pi\int_z^1 g_n(r)dr=z\int_z^1 \frac{f_n(p)T_n(p/z)dp}{(p^2-z^2)^{\frac{1}{2}}p} \qquad (17)$$

and, by differentiating with respect to z

$$g_n(r)=-\frac{1}{\pi}\frac{d}{dr}\int_r^1 \frac{rf_n(p)T_n(p/r)dp}{(p^2-r^2)^{\frac{1}{2}}p}. \qquad (18)$$

It has been shown that $g_n(r)$ determines $f_n(p)$ uniquely by Eq. (10), and it should now be shown that the above inversion formulae determine g_n uniquely. Suppose that there are two functions f_{n1} and f_{n2} which yield the same function g_n in Eq. (17), and let $f_{n1}-f_{n2}=F_n$. Then F_n satisfies the equation

$$\int_r^1 \frac{rT_n(p/r)F_n(p)dp}{(p^2-r^2)^{\frac{1}{2}}p}=0. \qquad (19)$$

In the range $r\leqslant p\leqslant 1$, $(p/r)\geqslant 1$, therefore, $T_n(p/r)\geqslant 1$. Hence, $T_n(p/r)(p^2-r^2)^{-\frac{1}{2}}p^{-1}\geqslant 0$ in the range of integration, and the only continuous solution of Eq. (19) is $F_n(p)=0$. Thus, Eq. (17) determines $g_n(r)$ uniquely. If Eq. (18) is examined for uniqueness, ignoring its origin in Eq. (17), the proof requires an extra step, caused by the additional operation of differentiation. For if f_{n1}, f_{n2}, and F_n have the same meaning as before, the equation which F_n must satisfy is

$$\int_r^1 \frac{rT_n(p/r)F_n(p)dp}{(p^2-r^2)^{\frac{1}{2}}p}=c=\text{constant}. \qquad (20)$$

Solutions of this exist which are not identically zero. For example, take $n=1$, $F_1=p(1-p^2)^{-\frac{1}{2}}$, then $c=(\pi/2)$. But this choice of F_1 violates the condition (c) of Sec. 3, which states that $|F_n(p)|\leqslant 2M_n(1-p^2)^{\frac{1}{2}}$ for $x_n\leqslant p\leqslant 1$, and it can be shown, by letting $r\to 1$, that if F_n satisfies this condition, c must be zero. Therefore, the above proof applies, $F_n=0$, and the solution is unique.

5. A GENERALIZATION

From a mathematical point of view, it is interesting to see whether the problem can be formulated and solved if the line integrals of a function are taken along families of curves other than straight lines. A formal solution has been found for a family of circles through the origin.

Let the circles be defined by two parameters (p,ϕ) and let the equation of the circles be

$$r=p\cos(\theta-\phi). \tag{21}$$

Then, if $f(p,\phi)$ is the line integral of $g(r,\theta)$ along the circle defined by (p,ϕ), one may proceed exactly as before with the Fourier analysis of both f and g and arrive at the set of integral equations

$$f_n(p)=2p\int_0^p \frac{g_n(r)T_n(r/p)dr}{(p^2-r^2)^{\frac{1}{2}}}. \tag{22}$$

Multiplying the left-hand side by $T_n(z/p)(z^2-p^2)^{-\frac{1}{2}}$ and integrating from $p=0$ to z, and changing the order of integration on the right, one gets

$$\int_0^z \frac{f_n(p)T_n(z/p)dp}{(z^2-p^2)^{\frac{1}{2}}} =2\int_0^z g_n(r)dr\int_r^z \frac{T_n(z/p)T_n(r/p)pdp}{(z^2-p^2)^{\frac{1}{2}}(p^2-r^2)^{\frac{1}{2}}}. \tag{23}$$

Making the substitution $t=rz/p$, and using Eq. (16),

$$\int_r^z \frac{T_n(z/p)T_n(r/p)pdp}{(z^2-p^2)^{\frac{1}{2}}(p^2-r^2)^{\frac{1}{2}}}=\frac{\pi}{2}. \tag{24}$$

Substitution of Eq. (24) into Eq. (23) and differentiating as before yields

$$g_n(r)=\frac{1}{\pi}\frac{d}{dr}\int_0^r \frac{f_n(p)T_n(r/p)dp}{(r^2-p^2)^{\frac{1}{2}}}. \tag{25}$$

This solution has not been investigated in any detail, but its existence leads one to think that such generalizations may be carried further.

6. AN EXPERIMENTAL TEST

An experiment was carried out in the simplest case where g was a function of r only. The specimen was a disk, 5 cm thick and 20 cm in diameter, made in the following way. A central cylinder of aluminum, 1.13 cm in diameter was surrounded by an aluminum annulus with an inner diameter of 1.13 cm and an outer diameter of 10.0 cm, and this in turn was surrounded with a wooden (oak) annulus with an inner diameter of 10.0 cm and an outer diameter of 20.0 cm. A peculiarity in the results lead to an investigation of the materials used, and it transpired that the central cylinder had been made of pure aluminum while the annulus had been made with an aluminum alloy. A 7-mCi Co^{60} source produced a gamma-ray beam which was collimated by a 15-cm lead shield with a circular hole in it. The gamma rays were detected by a Geiger counter which was well shielded and preceded by a second collimator. The gamma-ray beam had an over-all width of 7 mm. Because of the symmetry of the sample it was only necessary to measure $f(p,\phi)$ at one angle, and it was measured for $p=0$ cm to $p=12.5$ cm at 5-mm intervals. At least 20 000 counts were taken at each setting to reduce statistical counting errors to less than 1%, and the usual corrections for backgrounds and deadtime were made.

For this case, $(n=0)$, the solution (18) may be written

$$g_0(r)=-\frac{d}{dr}\left[\frac{r}{\pi}\int_r^1 \frac{f_0(p)dp}{(p^2-r^2)^{\frac{1}{2}}p}\right]=-\frac{dJ(r)}{dr}. \tag{26}$$

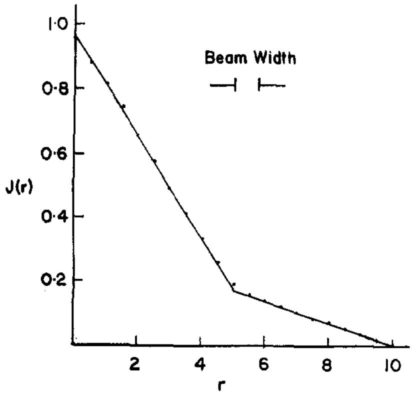

FIG. 1. $J(r)$ vs r in cm. Straight lines represent $J(r)$ calculated from measured values of the absorption coefficients, points give $J(r)$ calculated from Eq. (26).

The expression $J(r)$ was found from the experimentally determined $f_0(p)$ by numerical integration, except that an analytic approximation was used in evaluating the integral near the singularity at $p=r$. The values of $J(r)$ so found are shown as points in Fig. 1. The values of the absorption coefficients of the aluminum alloy and the wood were found to be 0.161 ± 0.002 cm^{-1} and 0.0340 ± 0.0005 cm^{-1}, respectively, and a value[4] of 0.150 cm^{-1} was assumed for the inner aluminum cylinder. $J(r)$ was calculated using these values and is shown by the straight lines in Fig. 1. The agreement is good. The full width of the gamma-ray beam is also shown in Fig. 1.

This experiment is a test of the method only in the simplest case, but it does indicate that the effects of beam width need not be too serious. More stringent tests

[4] C. M. Davisson and R. D. Evans, Phys. Rev. 81, 404 (1951).

with more complicated samples are needed and these are being undertaken.

ACKNOWLEDGMENTS

The author would like to thank Dr. J. M. Grieve of the Radiology Department of Groote Schuur Hospital, Cape Town, for pointing out the necessity for a solution of the absorption problem, and A. M. Koehler of the Harvard Cyclotron Laboratory for pointing out that the solution could be used in the coincidence problem. The partial support of this work by the United States Atomic Energy Commission is also acknowledged.

JOURNAL OF APPLIED PHYSICS VOLUME 34, NUMBER 9 SEPTEMBER 1963

Residual Temperatures of Shocked Copper*

JOHN W. TAYLOR
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico
(Received 1 April 1963)

A photoelectric technique is described for measuring the residual temperature of strongly shocked metals. Data for copper shocked to pressures ranging from 0.9 to 1.7 Mbars are in good agreement with published calculations of temperatures.

INTRODUCTION

THE methods which may be used to deduce a complete thermodynamic equation of state from shock-wave data which yield a single (P,V,E) locus have been discussed extensively by Rice, McQueen, and Walsh[1] and by Walsh and Christian[2]. The calculations may be done in more than one way but they are all based on certain assumptions. The most important of these are: the metal is in thermodynamic equilibrium behind a shock wave, the elastic limit at high pressures does not increase by orders of magnitude above its value at zero pressure, and the Mie–Grüneisen equation of state,

$$(P-P_s)V=\gamma(V)(E-E_s), \qquad (1)$$

is valid over the entire pressure–volume–energy surface of interest. The subscript s refers to any standard state. Calculations for most materials are made with the additional assumption that gamma is a function of volume only and is given by the Dugdale–MacDonald formula,

$$\gamma(V)=-(V/2)[\partial^2(PV^{\frac{2}{3}})\partial V^2/\partial(PV^{\frac{2}{3}})\partial V]-\tfrac{1}{3}. \qquad (2)$$

Gamma also can be shown to be equal at zero pressure to the limit $2S-1$ where, $S=(dU_s/dU_p)_{U_p=0}$. U_s is shock velocity and U_p is shock particle velocity. It is necessary to assume that the yield point does not depend markedly on pressure in order to be able to employ the theory of the isentropic flow of an ideal fluid in a consideration of release waves at the free surface of a shocked metal.

Recent work in which time-resolved measurements were made of the free surface velocity of shocked-annealed copper indicates that the elastic limit remains substantially constant up to 100 kbars (one kbar is 10^9 dyn/cm^2) at least and that shocks at pressures in excess of about 50 kbars in copper are propagated essentially as discontinuities.[3] Russian work[4] in which sound speeds were measured in copper at pressures in excess of 1000 kbar indicates that the high-pressure elastic limit is finite but still small and shows good agreement with sound speeds calculated with the Dugdale–MacDonald formula and Hugoniot data.[5] These results combined with the observation that at zero pressures the "dynamic" gamma $(2S-1)$ agrees very well with the "thermodynamic" gamma $V(\partial P/\partial T)_v/C_v$ lead one to suspect that copper is the metal most likely to conform to all the necessary assumptions and is, therefore, the most logical candidate for an experimental check of the theory.

The way of calculating temperatures chosen by Walsh and Christian was to assume that $\gamma/V=b$ is constant and use the equation,

$$T_H=T_0\exp\{b(V_0-V_H)\}+\exp\{-bV_H\} \times\left[\int_{V_0}^{V_H}\frac{f(V)\exp\{bV\}}{C_v}dV\right], \qquad (3)$$

where T_0, V_0 are temperature and volume under ambient conditions and T_H, V_H are the same quantities on the Hugoniot, and

$$f(V)=\tfrac{1}{2}[P_H+(V_0-V_H)(\partial P/\partial V)_H]. \qquad (4)$$

---

* Work done under the auspices of the U. S. Atomic Energy Commission.

[1] M. H. Rice, R. G. McQueen, and J. M. Walsh in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1958), Vol. VI.

[2] J. M. Walsh and R. H. Christian, Phys. Rev. **97**, 1544 (1955).

[3] J. W. Taylor and M. H. Rice (to be published).

[4] L. V. Al'tshuler, S. B. Kormer, A. A. Bakanova, and R. F. Trunin, Zh. Eksperim. i Teor. Fiz. **38**, 1061 (1960) [English transl.: Soviet Phys.—JETP **11**, 766 (1960)].

[5] R. G. McQueen and S. P. Marsh, J. Appl. Phys. **31**, 1253 (1960).